5.5 Problem Set

5.5.1 Questions on the Functional Field Integral

1. In chapter 4., the connection between the coherent state path integral and the Feynman path integral for a Harmonic oscillator was exposed. The aim of this problem is to extend this calculation to obtain the Lagrangian form of the path integral for a harmonic chain from the coherent state path integral. In chapter 1 we derived a free scalar field theory for the classical Harmonic chain. Qauntising the classical theory, we showed that, in second quantised form, the Hamiltonian was given by

$$\hat{H} = \sum_k \hbar\omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right),$$

where $\omega_k = k_s a^2 k^2/m$. Absorbing the zero-point energy into the definition of the functional integral measure $D[\bar{\psi}_k, \psi_k]$, the corresponding coherent state path integral for the quantum partition function assumes the form

$$\mathcal{Z} = \int D[\bar{\psi}_k, \psi_k] \exp\left[-\int_0^\beta d\tau \sum_k \left(\bar{\psi}_k \partial_\tau \psi_k + \hbar\omega_k \bar{\psi}_k \psi_k \right) \right],$$

where $\psi_k(\tau)$ denotes a complex scalar field, and the chemical potential has been set to zero $\mu = 0$.

(a) By applying the change of field variables,

$$\psi_k(\tau) = \left(\frac{m\omega_k}{2\hbar} \right)^{1/2} \left[q_k(\tau) + \frac{i p_k(\tau)}{m\omega_k} \right]$$

where we choose $\bar{q}_k(\tau) = q_{-k}(\tau)$ and $\bar{p}_k(\tau) = p_{-k}(\tau)$, show that the path integral can be cast in the Hamiltonian form of the Feynman path integral.

(b) Integrating over the fields $p_k(\tau)$, obtain the Lagrangian form of the path integral for the quantum partition function. Returning to the real space representation, show that the corresponding effective action takes the form

$$\mathcal{Z} = \int Dq(x) \exp\left[-\int_0^\beta d\tau \int_0^L dx \left(\frac{1}{2} m\dot{q}^2 + \frac{1}{2} k_s a^2 (\partial_x q)^2 \right) \right].$$

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2. **Zero temperature gap equation**: In our analysis of the field theory of the superconductor, by taking the order parameter Δ to be small, we restricted the validity of the field theory to vicinity of the critical temperature T_c. However, at zero temperature, we can obtain an estimate of the magnitude of the order parameter without resort to perturbative expansion.

In the mean-field approximation, i.e. taking Δ constant in space and time, show that the effective action for the BCS superconductor at *zero temperature* can be expressed in the form,[22]

$$\frac{S}{\beta L^d} = -\int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \operatorname{tr} \ln \begin{pmatrix} -i\omega + \epsilon_{\mathbf{k}} - \mu & \Delta \\ \Delta^* & -i\omega - \epsilon_{\mathbf{k}} + \mu \end{pmatrix} + \frac{|\Delta|^2}{g}.$$

Setting $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$, show that the action can be rewritten as

$$\frac{S}{\beta L^d} \simeq -\nu(\mu) \int d\xi \int \frac{d\omega}{2\pi} \ln\left(\omega^2 + |\Delta|^2 + \xi^2\right) + \frac{|\Delta|^2}{g}.$$

where $\nu(\mu)$ denotes the density of states at the Fermi level. Minimising the action with respect to Δ, obtain an expression for the gap equation. For $\omega_D \gg |\Delta|$, where ω_D denotes the Debye frequency cut-off on the ω integral, show that

$$|\Delta| = 2\omega_D \exp\left[-\frac{1}{g\nu(\mu)}\right].$$

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3. **Mean-Field Theory of the BCS Hamiltonian**: This question involves the investigation of the BCS Hamiltonian within the so-called mean-field approximation. It is also serves as useful revision of methods of second quantisation.

   In the mean-field approximation, the microscopic Bogoluibov Hamiltonian can be expressed in second quantised form

$$\hat{H} - \mu\hat{N} = \sum_{\mathbf{k}} \left[\sum_{\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \left(\Delta^* c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + \Delta c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}\right)\right] + \frac{|\Delta|^2}{g},$$

   where $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$ and the complex order parameter is uniform and taken to be determined self-consistently as $\Delta = g\sum_{\mathbf{k}} \langle \text{g.s.}| c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} |\text{g.s.}\rangle$ and $\Delta^* = g\sum_{\mathbf{k}} \langle \text{g.s.}| c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} |\text{g.s.}\rangle$. In the **Nambu spinor representation**

$$\Psi^\dagger_{\mathbf{k}} = \begin{pmatrix} c^\dagger_{\mathbf{k}\uparrow} & c_{-\mathbf{k}\downarrow} \end{pmatrix}, \qquad \Psi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix},$$

   the Hamiltonian can be recast in the form

$$\hat{H} = \sum_{\mathbf{k}} \left[\Psi^\dagger_{\mathbf{k}} \begin{pmatrix} \xi_{\mathbf{k}} & -\Delta \\ -\Delta^* & -\xi_{\mathbf{k}} \end{pmatrix} \Psi_{\mathbf{k}} + \xi_{\mathbf{k}}\right] + \frac{|\Delta|^2}{g}.$$

   (a) *Taking the order parameter Δ to be real*, show that the quadratic Hamiltonian can be diagonalised by a *unitary* transformation and gives

$$\hat{H} = \sum_{\mathbf{k}\sigma} \lambda_{\mathbf{k}} \alpha^\dagger_{\mathbf{k}\sigma} \alpha_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} (\xi_{\mathbf{k}} - \lambda_k) + \frac{\Delta^2}{g},$$

[22] Note that $\frac{1}{L^d}\sum_{\mathbf{k}} \overset{L\to\infty}{=} \int \frac{d^dk}{(2\pi)^d}$ and $\frac{1}{\beta}\sum_{\omega_m} \overset{\beta\to\infty}{=} \int \frac{d\omega}{2\pi}$

where

$$\begin{pmatrix} \alpha^\dagger_{\mathbf{k}\uparrow} \\ \alpha_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathbf{k}} & \sin\theta_{\mathbf{k}} \\ \sin\theta_{\mathbf{k}} & -\cos\theta_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c^\dagger_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow} \end{pmatrix}$$

and $\lambda_{\mathbf{k}} = (\xi_{\mathbf{k}}^2 + \Delta^2)^{1/2}$.

The elementary excitations of the superconducting system are known as **Bogoluibons** and are created by the operators $\alpha^\dagger_{\mathbf{k}\uparrow}$ and $\alpha^\dagger_{-\mathbf{k}\downarrow}$. The result above show that the quasi-particles have a minimum energy gap of Δ.

(b) In terms of quasi-particle operators, the ground state of the system can be written in the form $|\text{g.s.}\rangle = \prod_{\mathbf{k}} \alpha_{\mathbf{k}\uparrow}\alpha_{-\mathbf{k}\downarrow}|\Omega\rangle$, where $|\Omega\rangle$ represents the vacuum state — i.e. $|\text{g.s.}\rangle$ is annihilated by all quasi-particle operators $\alpha_{\mathbf{k}\uparrow}$ and $\alpha_{-\mathbf{k}\downarrow}$. Expanding, show that, up to a constant factor,

$$|\text{g.s.}\rangle = \prod_{\mathbf{k}} \alpha_{-\mathbf{k}\downarrow}\alpha_{\mathbf{k}\uparrow}|\Omega\rangle \sim \prod_{\mathbf{k}} \left(\cos\theta_{\mathbf{k}} - \sin\theta_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}\right)|\Omega\rangle.$$

With this choice, confirm that this state is normalised. Obtain an expression for the ground state energy.

(c) Expanding the bare electron operators in terms of the quasi-particle operators, or Bogoluibons, show that the self-consistency equation $\Delta = g\sum_{\mathbf{k}}\langle\text{g.s.}|c_{\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}|\text{g.s.}\rangle$ results in a self-consistent equation for Δ — the BCS Gap equation.