Quantum Condensed Matter Field Theory

Preface

The aim of this course is to provide a self-contained introduction to the basic tools and concepts of many-body quantum mechanics and quantum field theory, motivated by physical applications, and including the methods of second quantisation, the Feynman path integral and functional field integral. The course synopsis is outlined below. Items indicated by a † will either be covered in lectures (depending on time) or will be used as additional source material for problem sets and supervision. The italicised items represent particular mathematical concepts:

- ▷ COLLECTIVE EXCITATIONS: FROM PARTICLES TO FIELDS: Linear harmonic chain and free scalar field theory; *functional analysis*; quantisation of the classical field; phonons; †relation to quantum electrodynamics; concepts of broken symmetry, collective modes, elementary excitations and universality. [3]

- ▷ SECOND QUANTISATION: Fock states; creation and annihilation operators for bosons and fermions; representations of one- and two-body operators; *canonical transformations*; Applications to phonons; the interacting electron gas; Wannier states, strong correlation and the Mott transition; quantum magnetism and spin wave theory; *spin representations*; †spin liquids; the weakly interacting Bose gas. [6]

- ▷ PATH INTEGRAL METHODS: Propagators and construction of the Feynman Path integral; *Gaussian functional integration and saddle-point analyses*; relation to semi-classics and statistical mechanics; harmonic oscillator and the single well; double well, instantons, and tunneling; †metastability and the fate of the false vacuum. [7]

- ▷ MANY-BODY FIELD INTEGRAL: Bose and Fermi coherent states; *Grassmann algebra*; coherent state Path integral; quantum partition function; Applications to Bogoluibov theory of the weakly interacting Bose gas and superfluidity; Cooper instability and the BCS condensate; Ginzburg-Landau phenomenology and the connection to classical statistical field theory, †Gauge theory and the Anderson-Higgs mechanism; †Resonance superfluidity in ultracold atomic gases and the BEC to BCS crossover, †Peierls instability. [8]

Course Objectives

From analytical dynamics and fluid mechanics, to electrodynamics and quantum mechanics, lectures can often leave an impression that to each problem in physics a specific and formal exact solution is at hand. Such misconceptions are often reinforced by the allure of sophisticated analytical machinery developed in courses devoted to mathematical methods. However, the limitations of a 'first-principles' or 'microscopic approach' is nowhere more exposed than in the study of strongly interacting classical and quantum *many-particle* systems. The aim of this course is to introduce modern methods of theoretical physics tailored to the description of collective phenomena where microscopic (and, often, perturbative) approaches fail. The fundamental concepts on which we rely are (broken) symmetries, collective modes, elementary excitations, and universality. The foundation of our approach will be functional methods of classical and quantum field theory.

To introduce the notion and significance of the quantum field, the first few introductory lectures involve the construction and quantisation of a classical continuum field theory starting from a discrete model of lattice vibrations. By the end of the course, we will see that this system provides a platform to describe the elementary excitations of spin-waves in a quantum antiferromagnet, excitations in a weakly interacting Bose gas, and the relativistic scalar field!

In the study of quantum many-body phenomena in both high energy and condensed matter physics, second quantisation provides a basic and common language. In the next few lectures, a formal introduction to this operator method is consolidated by applications to both fermionic and bosonic systems. Beginning with a study of the strongly interacting electron gas, we exploit the second quantisation to expose an instability towards the formation of an electron "solid phase" — out of which a magnetic state emerges. This application in turn motivates the investigation of the hydrodynamic or spin-wave spectrum of the quantum Heisenberg spin (anti)ferromagnet. We then close this section with a discussion of the weakly interacting dilute Bose gas.

As preparation for the field theory of the many-body system, the functional field integral method will be introduced and developed within the framework of the Feynman path integral. Emphasis will be given to the connection of the path integral to classical Lagrangian mechanics through the semi-classical expansion, as well as the relation to the quantum and classical statistical partition function through the Euclidean time action. The example of a single well and the instanton approach to the double well will be explored in lectures. Further applications to metastability and macroscopic quantum tunneling will be discussed depending on time.

In the study of both high energy and condensed matter physics, methods of quantum statistical field theory play a central role. Although modern field theory applications in the respective fields have developed to a high degree of specialisation, a common origin is shared. The aim of the remaining lectures is to introduce the subject of quantum and statistical field theory placing emphasis on generic concepts. Introducing the bosonic and fermionic coherent state, the first two lectures are concerned with the microscopic derivation of the coherent state path integral. The latter is applied to the weakly interacting Bose gas and the phenomenon of superfluidity. Continuing this theme, we then explore the pair instability of the interacting electron gas and the formation of the superconducting BCS condensate. Here, the connection between the effective BCS action and the Ginzburg-Landau theory of phase transitions and critical phenomena will be emphasised.

Problem Sets

The Problem sets represent an integral part of the course providing the means to reinforce key ideas as well as practice techniques. Problems indicated by a † symbol are regarded as particularly challenging.

Books

Several texts cover the introduction to second quantisation, path integrals and quantum field theory. However, one can draw great benefit by studying a variety of different texts. The bibliography below includes many books, some explicitly referenced in these lecture notes, others that I have found useful in preparing the course, and still others that are frequently mentioned but which I find less useful. A note has been included concerning their relevance and accessibility. Those books which would seem to be of particular use have been denoted by a "*". You will also, no doubt, find books *not* included in the this list which are both relevant and useful...

Alongside the literature, detailed lecture notes have been prepared to supplement the course. Although these lecture notes will include additional commentaries and examples not covered in the course, they are intended to complement the material contained in the lectures, and they *will not* form part of the examinable material for the course.

Bibliography

[1] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, **Methods of Quantum Field Theory in Statistical Physics**, Dover Publications, Inc., 1975. *Advanced, historically significant, but somewhat out of date (only diagrammatics), and often overly concise.*

[2] A. Altland and B. D. Simons, **Condensed Matter Field Theory**, CUP Second Edition (2010). *Originally inspired by the preparation of this course, this text includes material that reaches far beyond these lectures.*

[3] N. W. Ashcroft and N. D. Mermin, **Solid State Physics**, Holt-Saunders International Eds. (1983). *Classical reference on solid state physics.*

[4] A. Auerbach, **Interacting Electrons and Quantum Magnetism**, Spinger-Verlag, New York (1994). *Tutorial discussion on applications of second quantisation and the path integral to problems in quantum magnetism.*

[5] *P. W. Anderson, **Concepts in Solids: Lectures of the Theory of Solids**, World-Scientific, Singapore (1997). *Exellent introduction to the concepts of elementary excitations in condensed matter physics.*

[6] P. W. Anderson, **Basic Notions in Condensed Matter Physics**, Benjamin (1984). *Inspirational but not tutorial — a highly subjective view of modern condensed matter physics from one of the most influential figures.*

[7] S. Coleman, **Aspects in Symmetry — Selected Erice Lectures** (CUP), 1985. *Chapter 5. has an excellent introduction to the subject of instantons — including applications to particle physics and cosmology — highly recommended.*

[8] A. Das, **Field Theory: A Path Integral Approach**, World Scientific Publishing, (1993). *A good, cheap, introduction to the path integral approach.*

[9] R. P. Feynman and A. R. Hibbs, **Quantum Mechanics and Path Integrals**, McGraw-Hill, New York, (1965). *A classic text: inevitably excludes a lot of modern material but still an excellent introduction to the Feynman path integral.*

[10] R. P. Feynman, **Statistical Mechanics**, Benjamin, New York, (1972). *Another classic text.*

[11] C. Kittel, **Quantum Theory of Solids**, Wiley (1963). *Not to be confused with the other famous text by the same author — good introduction to second quantisation and the physics of the weakly interacting electron gas.*

[12] *N. Nagaosa, **Quantum Field Theory in Condensed Matter Physics**, Sringer 1999. *An excellent and useful modern text. Short on detail but plenty of examples taken from condensed matter physics. See also the second volume in the series.*

[13] *J. W. Negele and H. Orland, **Quantum Many-Particle Systems**, Addison-Wesley Publishing, 1988. *Covers several core topics in this course including second quantisation and path integrals — high on detail, short on examples.*

[14] D. Pines and P. Nozières, **The Theory of Quantum Liquids — Neutral Fermi Liquids**, Addison-Wesley Publishing, NY (1989). *A classic text on solid state physics. Volume 2 is also interesting and relevant.*

[15] V. N. Popov, **Functional Integrals and Collective Excitations**, CUP, (1987). *Advanced and concise introduction to path integral methods generally in condensed matter physics — useful for much of the course.*

[16] *L. S. Schulman, **Techniques and Applications of Path Integration**, John Wiley & Sons, 1981. *Excellent introduction to the Feynman path integral.*

[17] A. M. Tsvelik, **Quantum Field Theory in Condensed Matter Physics**, (CUP) 1995. *A modern and entertaining discussion of applications of quantum field theory in condensed matter physics. Advanced, but useful as a source of inspiration.*

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