Lecture XXIV: †Superconductivity and Gauge Invariance

To establish origin of perfect diamagnetism and zero resistance,
one must accommodate electromagnetic field in Ginzburg-Landau Action

▷ Inclusion of electromagnetic field into BCS action: $\hat{\mathbf{p}} \to \hat{\mathbf{p}} - e\mathbf{A}$ $(c = 1)$

$$\mathcal{L}_{\rm EM} = -F_{\mu\nu}F^{\mu\nu}/4, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Repitition of field theory in presence of vector field obtains
generalised Ginzburg-Landau theory: $\mathcal{Z} = \int D\mathbf{A} \int D[\Delta, \bar{\Delta}]e^{-S}$

$$S = \beta \int d\mathbf{r}\Big[\frac{t}{2}|\Delta|^2 + \frac{K}{2}|(\partial + i2e\mathbf{A})\Delta|^2 + u|\Delta|^4 + \overbrace{\frac{1}{2}(\partial \times \mathbf{A})^2}^{\mathcal{L}_{\rm EM}}\Big]$$

focusing only on spatial fluctuations of $\mathbf{A}$

▷ Gauge Invariance: Action invariant under local gauge transformation

$$\mathbf{A} \mapsto \mathbf{A}' = \mathbf{A} - \partial\phi(\mathbf{r}), \qquad \Delta \mapsto \Delta' = e^{-2ie\phi(\mathbf{r})}\Delta$$

$$(\partial + i2e\mathbf{A})\Delta \mapsto (\partial + i2e(\mathbf{A} - \partial\phi))e^{-2ie\phi(\mathbf{r})}\Delta = e^{-2ie\phi(\mathbf{r})}(\partial + i2e\mathbf{A})\Delta$$

i.e. $|(\partial + i2e\mathbf{A})\Delta|^2$ (as well as $\partial \times \mathbf{A}$) invariant

▷ Anderson-Higgs mechanism:
phase of complex order parameter $\Delta = |\Delta|e^{-2ie\phi(\mathbf{r})}$ absorbed into $\mathbf{A} \mapsto \mathbf{A}' = \mathbf{A} - \partial\phi(\mathbf{r})$

$$S = \beta \int d\mathbf{r}\left[\frac{t}{2}|\Delta|^2 + \frac{K}{2}(\partial|\Delta|)^2 - \frac{m_\nu^2}{2}\mathbf{A}^2 + u|\Delta|^4 + \frac{1}{2}(\partial \times \mathbf{A})^2\right]$$

where $m_\nu^2 = 4e^2K|\Delta|^2$

i.e. massless phase degrees of freedom $\phi(\mathbf{r})$ have disappeared!
and photon field $\mathbf{A}$ has acquired a ‘mass’!

Example of a general principle:

“Below T_c, Goldstone bosons and the gauge field conspire to create massive excitations,
and the massless excitations are unobservable”, cf. electroweak theory

▷ Meissner effect: minimisation of action w.r.t. $\mathbf{A}$

$$\partial \times \overbrace{(\partial \times \mathbf{A})}^{\mathbf{B}} + m_\nu^2\mathbf{A} = 0 \qquad \mapsto \qquad (\partial^2 - m_\nu^2)\mathbf{B} = 0$$

$\mathbf{B} = 0$ is the only constant uniform solution $\rightsquigarrow$ perfect diamagnetism

Free energy of superconductor first proposed on phenomenological grounds — how?
...& why is crude gradient expansion so successful?

▷ Statistical Field Theory: Ferromagnetism Revisited

Superconducting phase transition is an example of a critical phenomena

Close to the critical point, the thermodynamic properties of a system are dictated by 'universal' characteristics

To understand why, consider simpler prototype system: the classical Ising ferromagnet:

$$\beta H = -J \sum_{\langle ij \rangle} S_i^z S_j^z + H \sum_i S_i^z, \qquad S_i^z = \pm 1$$

Equilibrium Phase diagram?
What happens in the vicinity of critical point?

(1) First order transition — order parameter (magnetisation) changes discontinuously
correlation length remains finite

(2) Second order transition — order parameter changes continuously
correlation length diverges

...motivates consideration of "hydrodynamic" theory of classical partition function

$$\mathcal{Z} = e^{-\beta F} = \int DS(\mathbf{r})\; e^{-\beta H[S(\mathbf{r})]}$$

βH constrained (only) by symmetry (translation, rotation, etc.)

$$\beta H[S(\mathbf{r})] = \int d\mathbf{r} \left[\frac{t}{2} S^2 + \frac{K}{2} (\partial S)^2 + u S^4 + \cdots + BS \right]$$

cf. Ginzburg-Landau Theory of superconductor

Landau theory: $S(\mathbf{r}) = S$ const.

$$\frac{F(S)}{L^d} = \frac{t}{2} S^2 + u S^4$$

▷ Generally second order phase transitions divide into Universality classes
with the same characteristic critical behaviour

E.g. (1) Ising model — liquid/gas: $S \to$ density ρ, $H \to$ pressure P

E.g. (2) Superconductivity — classical XY ferromagnet

subject of statistical field theory...