Lecture XIV: Coherent States

Discuss element of PI construction which demands generalisation

▷ Generalisation of FPI to many-body systems problematic
due to particle indistinguishability and statistics

Can second quantisation help? *automatically respects particle statistics*

Require complete basis on the Fock space to construct PI

Such eigenstates exist and are known as Coherent States
reference: Negele and Orland

▷ Coherent States (Bosons)

What are eigenstates of Fock space operators: a_i and $a_i^\dagger$ s.t. $[a_i, a_j^\dagger] = \delta_{ij}$?

Being a state of the Fock space, an eigenstate $|\phi\rangle$ can be expanded as

$$|\phi\rangle = \sum_{n_1,n_2,\cdots} C_{n_1,n_2,\cdots} \frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1}} \frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2}} \cdots |0\rangle$$

N.B. notation $|0\rangle$ for vacuum state!

(i) $a_i^\dagger|\phi\rangle = \phi_i|\phi\rangle$? — in fact, eigenstate of $a_i^\dagger$ can not exist:
if the minimum occupation of $|\phi\rangle$ is n_0, the minimum of $a_i^\dagger|\phi\rangle$ is $n_0 + 1$

(ii) $a_i|\phi\rangle = \phi_i|\phi\rangle$? — can exist and given by: $\boxed{|\phi\rangle \equiv \exp[\sum_i \phi_i a_i^\dagger]|0\rangle}$ N.B. $\phi \equiv \{\phi_i\}$

Proof: *since a_i commutes with all $a_j^\dagger$ for $j \neq i$ — focus on one element i*

$$a \exp(\phi a^\dagger)|0\rangle = [a, \exp(\phi a^\dagger)]|0\rangle =$$
$$= \sum_{n=0}^{\infty} \frac{\phi^n}{n!}[a, (a^\dagger)^n]|0\rangle = \sum_{n=1}^{\infty} \frac{n\phi^n}{n!}(a^\dagger)^{n-1}|0\rangle = \phi \exp(\phi a^\dagger)|0\rangle$$

$$a(a^\dagger)^n = aa^\dagger(a^\dagger)^{n-1} = (1 + a^\dagger a)(a^\dagger)^{n-1} = (a^\dagger)^{n-1} + a^\dagger a(a^\dagger)^{n-1} = n(a^\dagger)^{n-1} + (a^\dagger)^n a$$

i.e. $|\phi\rangle$ is eigenstate of all a_i with eigenvalue ϕ_i — known as Bosonic coherent state

▷ Properties of coherent state:

- Hermitian conjugation:

$$\forall i : \quad \langle\phi|a_i^\dagger = \langle\phi|\bar{\phi}_i$$

$\bar{\phi}_i$ is complex conjugate of ϕ_i

- By direct application of ∂_{ϕ_i} (and operator commutativity):

$$\forall i: \quad a_i^\dagger |\phi\rangle = \partial_{\phi_i} |\phi\rangle$$

- Overlap: with $\langle\theta| = (|\theta\rangle)^\dagger = \langle 0| e^{\sum_i \bar{\theta}_i a_i}$

$$\langle\theta|\phi\rangle = \langle 0| e^{\sum_i \bar{\theta}_i a_i} |\phi\rangle = e^{\sum_i \bar{\theta}_i \phi_i} \langle 0|\phi\rangle = \exp\left[\sum_i \bar{\theta}_i \phi_i\right]$$

i.e. states are not orthogonal! *operators not Hermitian*

- Norm: $\langle\phi|\phi\rangle = \exp\left[\sum_i \bar{\phi}_i \phi_i\right]$

- Completeness — resolution of id. (for proof see notes)

$$\boxed{\int \prod_i \frac{d\bar{\phi}_i d\phi_i}{\pi} e^{-\sum_i \bar{\phi}_i \phi_i} |\phi\rangle\langle\phi| = \mathbf{1}_{\mathcal{F}}}$$

where $d\bar{\phi}_i d\phi_i = d\mathrm{Re}\,\phi_i d\mathrm{Im}\,\phi_i$

▷ Coherent States (Fermions)

Following bosonic case, seek state $|\eta\rangle$ s.t.

$$a_i|\eta\rangle = \eta_i|\eta\rangle, \qquad \eta = \{\eta_i\}$$

But anticommutativity $[a_i, a_j]_+ = 0$ $(i \neq j)$ implies eigenvalues η_i anticommute!!

$$\eta_i \eta_j = -\eta_j \eta_i$$

η_i can not be ordinary numbers — in fact, they obey...

▷ Grassmann Algebra

In addition to anticommutativity, defining properties:

(i) $\eta_i^2 = 0$ (cf. fermions) but note: these are not operators, i.e. $[\eta_i, \bar{\eta}_i]_+ \neq 1$

(ii) Elements η_i can be added to and multiplied by ordinary complex numbers

$$c + c_i \eta_i + c_j \eta_j, \quad c_i, c_j \in \mathcal{C}$$

(iii) Grassmann numbers anticommute with fermionic creation/annihilation operators

$$[\eta_i, a_j]_+ = 0$$

▷ Calculus of Grassmann variables:

(iv) Differentiation: $\partial_{\eta_i} \eta_j = \delta_{ij}$

N.B. ordering $\partial_{\eta_i} \eta_j \eta_i = -\eta_j \partial_{\eta_i} \eta_i = -\eta_j$ for $i \neq j$

(v) Integration: $\int d\eta_i = 0$, $\int d\eta_i \eta_i = 1$

i.e. differentiation and integration have the same effect!!

▷ Gaussian integration:

$$\int d\bar{\eta} d\eta \, e^{-\bar{\eta} a \eta} = \int d\bar{\eta} d\eta \, (1 - \bar{\eta} a \eta) = a \int d\bar{\eta} \bar{\eta} \int d\eta \eta = a$$

$$\int d\bar{\eta} d\eta \, e^{-\bar{\eta}^T \mathbf{A} \eta} = \det \mathbf{A} \qquad \text{(exercise)}$$

cf. ordinary complex variables

▷ Functions of Grassmann variables:

Taylor expansion terminates at low order since $\eta^2 = 0$, e.g.

$$F(\eta) = F(0) + \eta F'(0)$$

Using rules

$$\int d\eta F(\eta) = \int d\eta \, [F(0) + \eta F'(0)] = F'(0) \equiv \partial_\eta F[\eta]$$

i.e. differentiation and integration have same effect on $F[\eta]$!

Usually, one has a function of many variables $F[\eta]$, say $\eta = \{\eta_1, \cdots \eta_N\}$

$$F(\eta) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n F(0)}{\partial \eta_i \cdots \partial \eta_j} \eta_j \cdots \eta_i$$

but series must terminate at $n = N$

with these preliminaries we are in a position to introduce the

▷ Fermionic coherent state: $\boxed{|\eta\rangle = \exp[-\sum_i \eta_i a_i^\dagger]|0\rangle}$ i.e. $\eta = \{\eta_i\}$

Proof (cf. bosonic case)

$$a \exp(-\eta a^\dagger)|0\rangle = a(1 - \eta a^\dagger)|0\rangle = \eta a a^\dagger |0\rangle = \eta|0\rangle = \eta \exp(-\eta a^\dagger)|0\rangle$$

Other defining properties mirror bosonic CS — *problem set*

▷ Differences:

(i) Adjoint: $\langle \eta | = \langle 0 | e^{-\sum_i a_i \bar{\eta}_i} \equiv \langle 0 | e^{\sum_i \bar{\eta}_i a_i}$ but N.B. $\bar{\eta}_i$ not related to η_i!

(ii) Gaussian integration: $\displaystyle\int d\bar{\eta} d\eta \, e^{-\bar{\eta}\eta} = 1$ N.B. no π's

Completeness relation

$$\boxed{\int \prod_i d\bar{\eta}_i d\eta_i e^{-\sum_i \bar{\eta}_i \eta_i} |\eta\rangle\langle\eta| = \mathbf{1}_F}$$