Lecture X: Feynman Path Integral

Although the second quantisation provides a convenient formulation of many-body systems, it admits solution only for systems that are effectively free. In our choice of applications, we were careful to consider only those systems for which interaction effects could be considered as small, e.g. large spin in quantum magnetism of weak interaction in the dilute Bose gas. Yet interactions can have a profound effect leading to transitions to new phases with elementary excitations very different from the bare particles. To address such phenomena, it is necessary to switch to a new formulation of quantum mechanics. However, to do so, it will be necessary to leave behind many-body theories and return to single-particle systems.

▷ MOTIVATION:

- Alternative formulation of QM (cf. canonical quantisation)
- Close to classical construction — i.e. semi-classics easily retrieved
- Effective formulation of non-perturbative approaches
- Prototype of higher-dimensional field theories

▷ TIME-DEPENDENT SCHRÖDINGER EQUATION

$$\boxed{i\hbar\partial_t|\Psi\rangle = \hat{H}|\Psi\rangle}$$

$$\text{Formal solution: } |\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = \sum_n e^{-iE_nt/\hbar}|n\rangle\langle n|\psi(0)\rangle$$

▷ Time-evolution operator

$$|\Psi(t')\rangle = \hat{U}(t',t)|\Psi(t)\rangle, \qquad \hat{U}(t',t) = e^{-\frac{i}{\hbar}\hat{H}(t'-t)}\theta(t'-t) \qquad \text{N.B. Causal}$$

- Real-space representation:

$$\Psi(q',t') \equiv \langle q'|\Psi(t')\rangle = \langle q'|\hat{U}(t',t) \overset{\int dq|q\rangle\langle q|}{\wedge} |\Psi(t)\rangle = \int dq\, U(q',t';q,t)\Psi(q,t),$$

where $U(q',t';q,t) = \langle q'|e^{-\frac{i}{\hbar}\hat{H}(t'-t)}|q\rangle\theta(t'-t)$ — propagator or Green function

$$\left(i\hbar\partial_{t'} - \hat{H}\right)\hat{U}(t'-t) = i\hbar\delta(t'-t) \qquad \text{N.B. } \partial_{t'}\theta(t'-t) = \delta(t'-t)$$

Physically: $U(q',t';q,t)$ describes probability amplitude for particle to propagate from q at time t to q' at time t'

▷ CONSTRUCTION OF PATH INTEGRAL

Feynman's idea: separate time evolution into $N \to \infty$ discrete time steps $\Delta t = t/N$

$$e^{-i\hat{H}t/\hbar} = [e^{-i\hat{H}\Delta t/\hbar}]^N$$

Then separate the operator content so that momentum operators stand to the left and position operators to the right:

$$e^{-i\hat{H}\Delta t/\hbar} = e^{-i\hat{T}\Delta t/\hbar} e^{-i\hat{V}\Delta t/\hbar} + O(\Delta t^2)$$

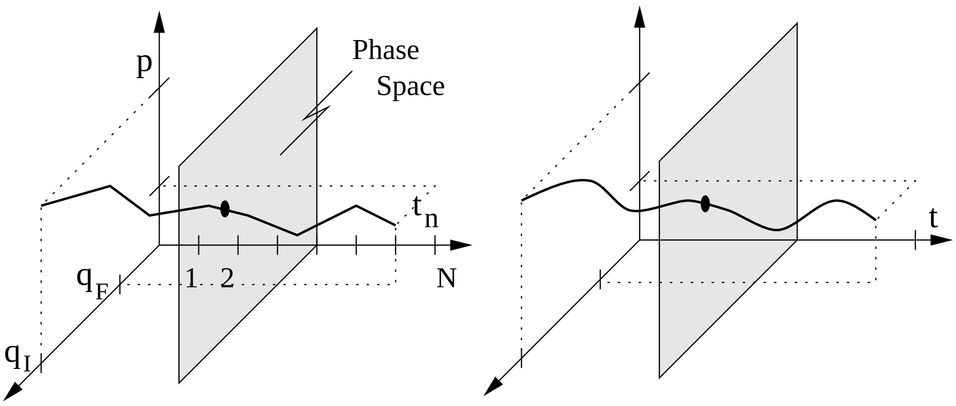
$$\langle q_F | [e^{-i\hat{H}\Delta t/\hbar}]^N | q_I \rangle \simeq \langle q_F |_{\wedge} e^{-i\hat{T}\Delta t/\hbar} e^{-i\hat{V}\Delta t/\hbar}{}_{\wedge} \cdots {}_{\wedge} e^{-i\hat{T}\Delta t/\hbar} e^{-i\hat{V}\Delta t/\hbar} | q_I \rangle$$

Inserting at $\wedge$ resol. of id. $= \int_{-\infty}^{\infty} dq_n \int_{-\infty}^{\infty} dp_n |q_n\rangle\langle q_n|p_n\rangle\langle p_n|$, and using $\langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{iqp/\hbar}$,

$$e^{-i\hat{V}\Delta t/\hbar} |q_n\rangle\langle q_n|p_n\rangle\langle p_n| e^{-i\hat{T}\Delta t/\hbar} = |q_n\rangle e^{-iV(q_n)\Delta t/\hbar} \langle q_n|p_n\rangle e^{-iT(p_n)\Delta t/\hbar} \langle p_n|,$$

and $\quad \langle p_{n+1}|q_n\rangle\langle q_n|p_n\rangle = \frac{1}{2\pi\hbar} e^{iq_n(p_n - p_{n+1})/\hbar}$

$$\langle q_F | e^{-i\hat{H}t/\hbar} | q_I \rangle = \int_{q_N = q_F, q_0 = q_I} \prod_{n=1}^{N-1} dq_n \prod_{n=1}^{N} \frac{dp_n}{2\pi\hbar} \exp\left[-\frac{i}{\hbar} \Delta t \sum_{n=0}^{N-1} \left(V(q_n) + T(p_{n+1}) - p_{n+1} \frac{q_{n+1} - q_n}{\Delta t} \right) \right]$$



i.e. at each time step, integration over the classical phase space coords. (q_n, p_n)

Contributions from trajectories where $(q_{n+1} - q_n)p_{n+1} > \hbar$ are negligible — motivates continuum limit

$$\langle q_F | e^{-i\hat{H}t/\hbar} | q_I \rangle = \overbrace{\int_{q_N = q_F, q_0 = q_I} \prod_{n=1}^{N-1} dq_n \prod_{n=1}^{N} \frac{dp_n}{2\pi\hbar}}^{\int_{q(t)=q_F, q(0)=q_I} D(q,p)} \exp\Big[-\frac{i}{\hbar} \overbrace{\Delta t \sum_{n=0}^{N-1}}^{\int_0^t dt'} \big(\overbrace{V(q_n) + T(p_{n+1})}^{H(q,p|_{t'=t_n})} - \overbrace{p_{n+1} \frac{q_{n+1} - q_n}{\Delta t}}^{p\dot{q}|_{t'=t_n}} \big) \Big]$$

▷ <u>Hamiltonian formulation of Feynman Path Integral:</u>

Propagator expressed as FUNCTIONAL INTEGRAL

$$\boxed{\langle q_F | e^{-i\hat{H}t/\hbar} | q_I \rangle = \int_{q(t)=q_F, q(0)=q_I} D(q,p) \exp\Big[\frac{i}{\hbar} \overbrace{\int_0^t dt' \overbrace{(p\dot{q} - H(p,q))}^{\text{Lagrangian}}}^{\text{Action}} \Big]}$$

Quantum transition amplitude expressed as sum over all possible phase space trajectories (subject to appropriate b.c.) and weighted by classical action

▷ Lagrangian formulation: for "free-particle" Hamiltonian $H(p,q) = p^2/2m + V(q)$

$$\langle q_F | e^{-i\hat{H}t/\hbar} | q_I \rangle = \int_{q(t)=q_F, q(0)=q_I} Dq \; e^{-(i/\hbar)\int_0^t dt' V(q)} \int Dp \, \overbrace{\exp\left[-\frac{i}{\hbar}\int_0^t dt' \left(\frac{p^2}{2m} - p\dot{q}\right)\right]}^{\text{Gaussian integral on p}}$$

$$\frac{p^2}{2m} - p\dot{q} \mapsto \frac{1}{2m}\overbrace{(p - m\dot{q})^2}^{p'^2} - \frac{1}{2}m\dot{q}^2$$

Functional integral justified by discretisation

$$\boxed{\langle q_F | e^{-i\hat{H}t/\hbar} | q_I \rangle = \int_{q(t)=q_F, q(0)=q_I} Dq \exp\left[\frac{i}{\hbar}\int_0^t dt' \left(\frac{m\dot{q}^2}{2} - V(q)\right)\right]}$$

$$Dq \equiv \lim_{N\to\infty} \left(\frac{Nm}{it2\pi\hbar}\right)^{N/2} \prod_{n=1}^{N-1} dq_n$$