Phase Transitions and Collective Phenomena

▷ Characterisation of complex systems?

Microscopic	$\overset{?}{\longrightarrow}$	Macroscopic
$\downarrow$		$\downarrow$
Microstates		Macrostate fns.
$\{p_i, q_i\}$, $\{\sigma_i\}$		P, V, T, S, E
$\downarrow$		$\downarrow$
Microscopic Hamiltonian (Classical/Quantum)		Laws of Thermodynamics

...How to establish connection?

▷ ...Statistical Mechanics: Partition Function

$$\mathcal{Z} = \sum_{\{\mu\}} e^{-\beta H[\{\mu\}]}$$

$$P(\{\mu\}) = \frac{e^{-H[\{\mu\}]/k_B T}}{\mathcal{Z}}$$

More is Different

▷ Non-Interacting System: $\mathcal{Z} = \mathcal{Z}_1 \mathcal{Z}_2 \cdots \mathcal{Z}_N$
$N \to \infty \rightsquigarrow$ Ideal Gas Laws

▷ Interactions: Singularities in $\mathcal{Z}$
$\rightsquigarrow$ Transitions to New Phases of Matter...

Classical Media	Solid-Liquid-Gas, etc.
Soft Cond. Matter	Liquid crystals, etc.
'Hard' Cond. Matter.	Superconductivity, Magnetism, etc.
High Energy Astrophysics	Baryogenesis, Higgs, etc.
Finance!?	No rich physicists yet...!

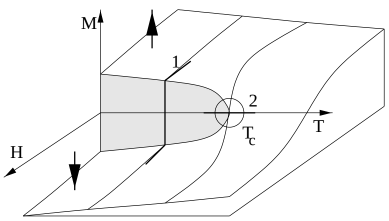
...Broken symmetries
$\rightsquigarrow$ characteristic Low-Energy Excitations:
Collective Modes

e.g. phonons, spin-waves, etc.

Phase Transitions

▷ Two important classes
c.f. phase diagram of classical Ising Ferromagnet

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j + HM, \qquad M = \sum_i \sigma_i$$



1. First order:
   Discontinuous change of 'order parameter' M

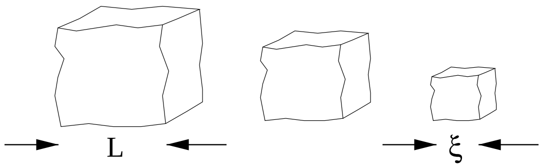
2. Second order:
   Order parmater grows continuously

Nature of Critical Point?

▷ In quantum/classical statistical mechanics
Second order phase transitions play special role...

...Why?

What happens to Correlation Length ξ?...



...i.e. scale at which macroscopic properties change

▷ At critical point T_c, correlation length ξ diverges...

$$\xi \sim |t|^{-\nu}, \qquad t = \frac{T - T_c}{T_c}$$

along with all other Thermodynamic quantities

Scale Invariance

▷ $\xi \to \infty$ — microscopic scales redundant at T_c!
$\rightsquigarrow$ coarse-grained description based only on fundamental symm., dimensionality, etc.

i.e. Ginzburg-Landau Phenomenology

Consequences:

1. Universality:

   Variations of Phenomenology limited...
   Phase transitions arranged in Universality classes

2. Scaling and Self-similarity :

   No characteristic length scale at the critical point...
   correlation functions scale invariant

3. Critical points heavily constrain "topology" of phase diagram

▷ How to explore/classify phases of matter in strongly interacting systems?

- Microscopic theories? $N \sim 10^{23}$!!
- Phenomenology for $\mathcal{Z}$ near Critical point?...

▷ Aim of course:

"to motivate, develop, & analyse critical phenomena in framework of Ginzburg-Landau phenomenology"

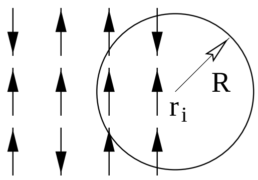
Approaches: Mean-Field Theory
Broken Symmetry & Fluctuations
Scaling Theory
Field Theory & Renormalisation Group

N.B. Close connection to QFT

Example: Classical Ising Ferromagnet

▷ Microscopic Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j, \qquad \{\sigma_i\} = \pm 1$$



▷ Coarse-grained Order parameter :

Local Magnetisation $m(\mathbf{r}_i) = \dfrac{1}{R^d} \displaystyle\sum_{|\mathbf{r}_j - \mathbf{r}_i| < R} \sigma_j$

▷ Ginzburg-Landau Phenomenology
coarse-grained Hamiltonian

$$\beta H = \int d^d\mathbf{r} \left[\frac{t}{2} m^2 + \frac{K}{2} (\nabla m)^2 + u m^4 + \cdots \right]$$

i.e. compatible with symmetries

Phenomenology

▷ Phase Transition (in dimensions $d > 1$) to Ferromagnetic state at $t = 0$

▷ Spontaneous Symmetry Breaking:

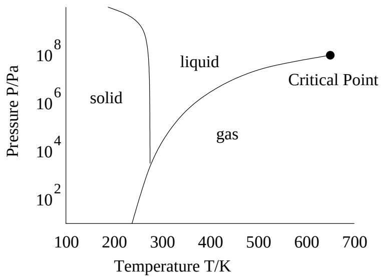
$$\begin{cases} \langle |M| \rangle_T = 0 & t > 0 \\ \langle |M| \rangle_T \sim |t|^\beta & t < 0 \end{cases}$$

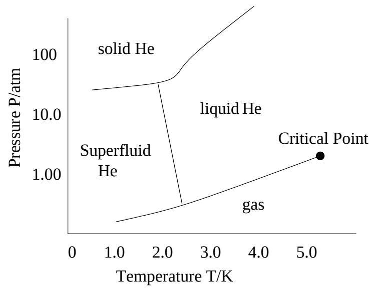
i.e. $t = \dfrac{T - T_c}{T_c}$

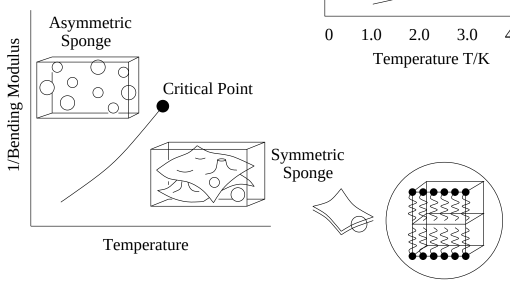
▷ Correlation length diverges as $\xi \sim t^{-\nu}$
$\chi \sim |t|^{-\gamma}$, $C \sim |t|^{-\alpha}$, $G(r) \sim |r|^{d-2+\eta}$, etc.

▷ Ising Universality Class:

- Uniaxial Ferro- and Antiferromagnet
- Liquid-Gas
- Lyotropic Liquid Crystal ('Sponge')



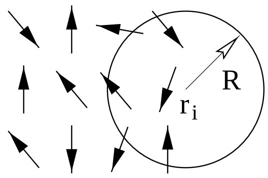




Example: Classical XY-Ferromagnet

▷ Microscopic Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j, \qquad \mathbf{S}_i = (\cos\theta_i, \sin\theta_i)$$



▷ Coarse-grained Order parameter:

Local Magnetisation $\mathbf{m}(\mathbf{r}_i) = \dfrac{1}{R^d} \displaystyle\sum_{|\mathbf{r}_j - \mathbf{r}_i| < R} \mathbf{S}_j$

▷ Ginzburg-Landau Phenomenology

coarse-grained Hamiltonian

$$\beta H = \int d^d\mathbf{r} \left[\frac{t}{2}\mathbf{m}^2 + \frac{K}{2}(\nabla \mathbf{m})^2 + u(\mathbf{m}\cdot\mathbf{m})^2 + \cdots \right]$$

i.e. compatible with symmetries

Phenomenology

▷ Phase Transition (in dimensions $d > 2$) to
Ferromagnetic state at $t = 0$

▷ Spontaneous Symmetry Breaking:

$$\begin{cases} \langle |\mathbf{m}| \rangle_T = 0 & t > 0 \\ \langle |\mathbf{m}| \rangle_T \sim |t|^\beta & t < 0 \end{cases}, \qquad \xi \sim t^{-\nu}$$

▷ Low-Energy Collective fluctuations — spin-waves

▷ High-Energy Vortex Configurations
$\rightsquigarrow$Topological Phase Transition in $d = 2$

▷ XY Universality Class

- Superconductors and Superfluids
- Melting in two-dimensions