Lecture XII: Bose-Einstein Condensation and Superfluidity

▷ Coherent State PI of quantum partition function:

$$\mathcal{Z} = \int_{\text{b.c.}} D(\bar{\psi}, \psi)\, e^{-S[\bar{\psi},\psi]},$$

with action

$$S = \int_0^\beta d\tau \int d\mathbf{r} \left[\bar{\psi}(\mathbf{r},\tau)(\partial_\tau + \hat{H}_0 - \mu)\psi(\mathbf{r},\tau) + \frac{g}{2}(\bar{\psi}(\mathbf{r},\tau)\psi(\mathbf{r},\tau))^2\right],$$

▷ Bose-Einstein Condensation

Consider non-interacting Bose gas:

$$\mathcal{Z}_0 \equiv \mathcal{Z}\Big|_{g=0} = \int \mathcal{D}(\bar{\psi}, \psi)\, \exp\left[-\sum_{an} \bar{\psi}_{an}\,(-i\omega_n + \epsilon_a - \mu)\,\psi_{an}\right].$$

where we assume $\epsilon_a \geq 0$ and $\epsilon_0 = 0$

Stability: chemical potential $\mu \geq 0$

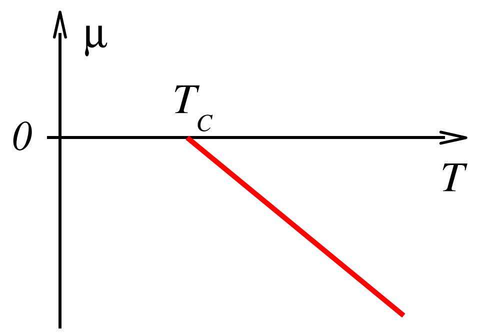
Number of particles:

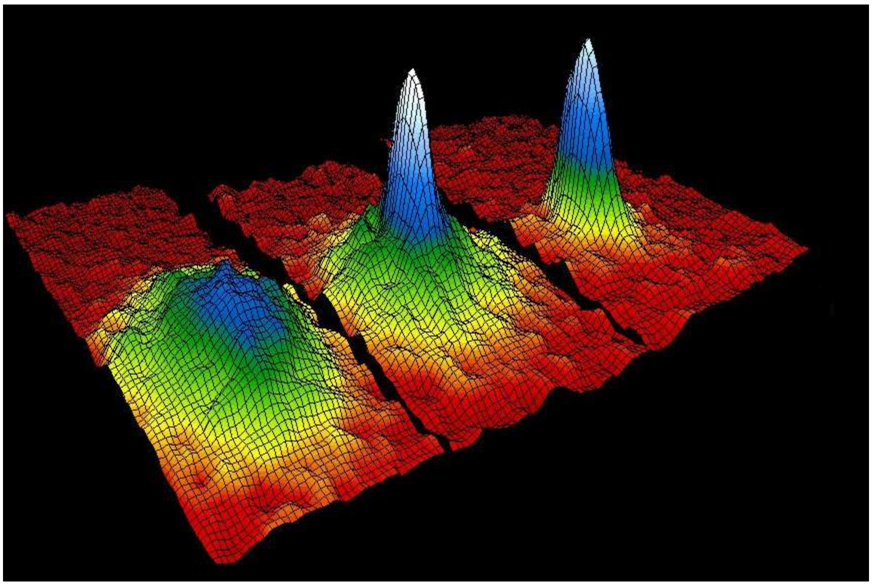
$$N(\mu) = T\sum_{na} \frac{1}{i\omega_n - \epsilon_a + \mu} = \sum_a n_{\text{B}}(\epsilon_a),$$

where $n_{\text{B}}(\epsilon) = (e^{\beta(\epsilon-\mu)} - 1)^{-1}$ denotes Bose distribution

At critical temperature T_c, μ reduces to zero. For $T < T_c$, μ remains zero and a microscopic number of particles $N - N_1$ must accumulate in the single-particle ground state: Bose-Einstein condensation

$$\sum_{a>0} n_{\text{B}}(\epsilon_a)\Big|_{\mu=0} \stackrel{T\leq T_c}{\equiv} N_1 < N$$





How can the phenomenon of BEC be implemented into the functional integral representation

characteristics of condensate described by the zero field component $\psi_0(\tau)$

But, below the condensation transition, its action apears unbound: both the chemical potential and the eigenvalue are zero. This means that the action of the zero Matsubara component $\psi_{0,0}$ vanishes.

Therefore, let us treat $\psi_0(\tau)$ as a *time-independent* Lagrange multiplier to be used to fix the number of particles below the T_c:

$$S_0|_{\mu=0^-} = -\bar{\psi}_0 \beta\mu\psi_0 + \sum_{a\neq 0,n} \bar{\psi}_{an}\left(i\omega_n + \epsilon_a - \mu\right)\psi_{an}\,,$$

With this definition,

$$N = -\partial_\mu \mathcal{F}_0|_{\mu=0^-} = T\partial_\mu \ln \mathcal{Z}_0|_{\mu=0^-} = \bar{\psi}_0\psi_0 + T\sum_{a\neq 0,n}\frac{1}{i\omega_n - \epsilon_a} = \bar{\psi}_0\psi_0 + N_1$$

i.e. $\bar{\psi}_0\psi_0 = N_0$ — no. particles in condensate

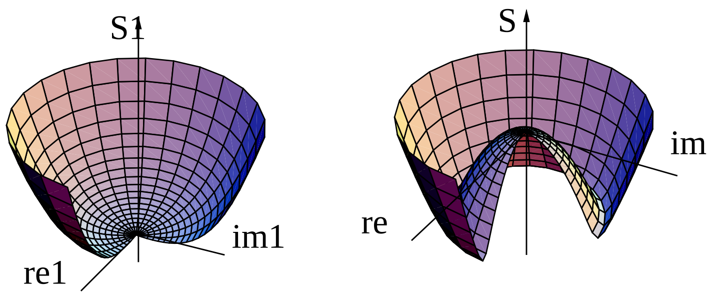
▷ Weakly Interacting Bose Gas

Dominant contirbution to $\mathcal{Z}$ from classical ψ_0 sector:

$$\frac{1}{\beta}S[\bar{\psi}_0,\psi_0] = -\mu\bar{\psi}_0\psi_0 + \frac{g}{2L^d}(\bar{\psi}_0\psi_0)^2$$

Saddle-point analysis:

$$\psi_0\left(-\mu + \frac{g}{L^d}\bar{\psi}_0\psi_0\right) = 0\,.$$



- ▷ For $\mu < 0$ (i.e. above the condensation threshold of the non-interacting system), the equation exhibits only the trivial solution $\psi_0 = 0$. This means that no stable condensate amplitude exists.
- ▷ Below the condensation threshold (i.e. for $\mu \geq 0$), the equation is solved by any configuration with $|\psi_0| = \gamma \equiv \sqrt{\mu L^d/g}$. Notice that $\bar{\psi}_0 \psi_0 \propto L^d$, reflecting the macroscopic population of the ground state.
- ▷ The equation couples only to the *modulus* of ψ_0. I.e. the solution of the stationary phase equation is continuously degenerate: Each configuration $\psi_0 = \gamma \exp(i\phi), \phi \in [0, 2\pi]$ is a solution. One ground state chosen $\rightsquigarrow$ spontaneous symmetry breaking.

▷ Superfluidity

Below T_c, expand action:

$$S[\bar{\psi}, \psi] = \int_0^\beta d\tau \int d\mathbf{r} \left[\bar{\psi}(\mathbf{r}, \tau) \left(\partial_\tau - \frac{\partial^2}{2m} - \mu \right) \psi(\mathbf{r}, \tau) + \frac{g}{2} (\bar{\psi}(\mathbf{r}, \tau)\psi(\mathbf{r}, \tau))^2 \right],$$

around mean-field $\bar{\psi}_0 = \psi_0 = (\mu L^d/g)^{1/2} = \gamma$

Fluctuations:

$$\psi(\mathbf{r}, \tau) = [\rho_0 + \delta\rho(\mathbf{r}, \tau)]^{1/2} e^{i\phi(\mathbf{r}, \tau)},$$
$$\bar{\psi}(\mathbf{r}, \tau) = [\rho_0 + \delta\rho(\mathbf{r}, \tau)]^{1/2} e^{-i\phi(\mathbf{r}, \tau)},$$

where $\rho_0 \equiv \gamma^2 = \bar{\psi}_0 \psi_0$ is the condensate density

cf. canonical transformation:

$$a(\mathbf{x}) \equiv \hat{\rho}(\mathbf{x})^{1/2} e^{i\hat{\phi}(\mathbf{x})}, \qquad a^\dagger(\mathbf{x}) \equiv e^{-i\hat{\phi}(\mathbf{x})} \hat{\rho}(\mathbf{x})^{1/2}$$

Assuming canonical commutation relations

$$[\hat{\rho}(\mathbf{x}), \hat{\phi}(\mathbf{x}')] = i\delta(x - x'), \qquad [\hat{\rho}(\mathbf{x}), \hat{\rho}(\mathbf{x}')] = [\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{x}')] = 0$$

confirm:

$$[a, a^\dagger] = \left[\hat{\rho}^{1/2} e^{i\hat{\phi}}, e^{-i\hat{\phi}} \hat{\rho}^{1/2} \right] = \hat{\rho} - \overbrace{e^{-i\hat{\phi}} \hat{\rho} e^{i\hat{\phi}}}^{e^{-i[\hat{\phi}, \,\cdot\,]}\hat{\rho}} = \hat{\rho} - \hat{\rho} + i[\hat{\phi}, \hat{\rho}] - \frac{1}{2}[\hat{\phi}, [\hat{\phi}, \hat{\rho}]] + \ldots = 1$$

Using density-phase relation

$$S[\delta\rho,\phi] \approx \int_0^\beta d\tau \int d\mathbf{r} \left[i\rho\partial_\tau\phi + \frac{\rho_0}{2m}(\partial\phi)^2 - \mu\rho\frac{g\rho_0^2}{2} \right]$$

First term has the canonical structure 'momentum $\times\, \partial_\tau$ (coordinate)' cf. canonically conjugate pair

Second term measures energy cost of spatially varying phase flucutations

Notice that $\phi(\mathbf{r},\tau) =$ const. does not incur an energy cost — i.e. ϕ is a Goldstone mode

Third term records the energy cost of massive fluctuations from potential minimum

Gaussian integration over $\delta\rho$ leads to Lagrangian formulation

$$S[\phi] \approx \frac{1}{2}\int_0^\beta d\tau \int d\mathbf{r} \left[\frac{1}{g}(\partial_\tau\phi)^2 + \frac{\rho_0}{m}(\partial\phi)^2 \right] .$$

cf. d-dimensional oscillator; $\omega_{\mathbf{k}} = |\mathbf{k}|\rho_0/mg$

Physical ramifications: quantum mechanical current operator

$$\hat{\mathbf{j}}(\mathbf{x},\tau) = \frac{i}{2m}\left[(\partial a^\dagger(\mathbf{x},\tau))a(\mathbf{x},\tau) - a^\dagger(\mathbf{x},\tau)\partial a(\mathbf{x},\tau) \right] \stackrel{\text{fun. int}}{\longrightarrow}$$
$$\longrightarrow \frac{i}{2m}\left[(\partial\bar\psi(\mathbf{x},\tau))\psi(\mathbf{x},\tau) - \bar\psi(\mathbf{x},\tau)\partial\psi(\mathbf{x},\tau) \right] \approx \frac{\rho_0}{m}\partial\phi(\mathbf{x},\tau)$$

i.e. $\partial\phi$ is measure of (super)current flow

Consider stationary phase equations

$$i\partial_\tau\phi = -g\delta\rho + \mu, \qquad i\partial_\tau\delta\rho = \frac{\rho_0}{m}\partial^2\phi = \partial\cdot\mathbf{j}$$

Second equation $\rightsquigarrow$ continuity equation

First equation: system adjusts to spatial fluctuations of density by dynamical phase fluctuation

Equations possess steady state solutions with non-vanishing current flow: setting $\partial_\tau\phi = \partial_\tau\delta\rho = 0$, obtain $\delta\rho = 0$ and $\partial\cdot\mathbf{j} = 0$, i.e. below the condensation temperature, a configuration with a uniform density profile can support a steady state divergenceless (super)current.

Notice that a 'mass term' in the ϕ action would spoil this property, i.e. within our present approach, the phenomenon of supercurrent flow is intimately linked to the Goldstone mode character of the ϕ field.

Steady state current flow in normal environments is prevented by the mechanism of energy dissipation, i.e. particles constituting the current flow scatter off imperfections *inside the system thereby converting part of their energy into the creation of elementary excitations. How can dissipative loss of energy be avoided. Trivially, no energy can be exchanged if there are no elementary excitations to create. In reality, this means that the excitations of the system are energetically high-lying such that the kinetic energy stored in the current-carrying particles is insufficient to create them. But this is not the situtation that we encounter in the superfluid! As we saw above, there is no energy gap separating the quasi-particle excitations of the system from the ground state. Rather, the dispersion* $\omega(\mathbf{k})$ *vanishes linearly as* $\mathbf{k} \to 0$. *However, there is an ingenuous argument due to Landau showing that a linear excitation spectrum indeed suffices to stabilize dissipationless transport:*

Consider flow of fluid through a pipe. Let us assume that the flow occurs at a uniform velocity $\mathbf{V}$. *Taking the mass (of a certain portion of the fluid) to be* M, *the current carries a total kinetic energy* $E_1 = M\mathbf{V}^2/2$. *Now, suppose we view the situation from the point of view of the fluid, i.e. we perform a Galileian transformation into its own rest frame. From the perspective of the fluid, the walls of the pipe appear as though they were moving with velocity* $-\mathbf{V}$. *Now, suppose that frictional forces between fluid and the wall lead to the creation of an elementary excitation of momentum* $\mathbf{p}$ *and energy* $\epsilon(\mathbf{p})$, *i.e. the fluid is no longer at rest but carries kinetic energy. After a Galileian transformation back to the laboratory frame we find that the energy of the fluid after the creation of the excitation is given by*

$$E_2 = \frac{M\mathbf{V}^2}{2} + \mathbf{p}\cdot\mathbf{V} + \epsilon(\mathbf{p})$$

Now, since all of the energy needed to manufacture the excitation must have been provided by the liquid itself, energy conservation requires that $E_1 = E_2$, *or* $-\mathbf{p}\cdot\mathbf{V} = \epsilon(\mathbf{p})$. *Since* $\mathbf{p}\cdot\mathbf{V} > -|\mathbf{p}||\mathbf{V}|$, *this condition can only be met if* $|\mathbf{p}||\mathbf{V}| > \epsilon(\mathbf{p})$. *While systems with a 'normal' gapless dispersion,* $\epsilon(\mathbf{p}) \sim \mathbf{p}^2$ *are compatible with this energy-balance relation (i.e. no matter how small* $|\mathbf{V}|$, *quasi-particles of low momentum can always be excited), both gapped dispersions* $\epsilon(\mathbf{p}) \stackrel{\mathbf{p}\to 0}{\longrightarrow} \text{const.}$ and *linear dispersions are incompatible if* $\mathbf{V}$ *becomes smaller than a certain* **critical velocity** V_*. *Specifically for a linear dispersion* $\epsilon(\mathbf{p}) = v|\mathbf{p}|$, *the critical velocity is given by* $V_* = v$. *For currents slower than that, the flow is necessarily dissipationless.*