Lecture IX: BCS Superconductivity

▷ Recall: We have seen that phonon exchange $\rightsquigarrow$ a pairing interaction which renders a single pair of electrons unstable towards the formation of a bound state (Cooper)

Motivated by this consideration, we have proposed a many-body generalisation of the pair state in the form of the variational BCS state

$$|\psi\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0\rangle$$

in which the anomalous average operator $c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}$ acquires a non-zero expectation value

Therefore, since we expect fluctuations in the pair operator $c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}$ to be small, we may set

$$c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} = b^*_{\mathbf{k}} + \overbrace{c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} - b^*_{\mathbf{k}}}^{\text{small}}$$

where $b^*_{\mathbf{k}} = \langle\psi|c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}|\psi\rangle$ so that the BCS Hamiltonian...

$$\hat{H} - \mu\hat{N} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - V \sum_{\mathbf{k}\mathbf{k}'} c^\dagger_{\mathbf{k}'\uparrow} c^\dagger_{-\mathbf{k}'\downarrow} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}, \qquad \xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$$

can be expanded as

$$\hat{H} - \mu\hat{N} \simeq \sum_{\mathbf{k}} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - V \sum_{\mathbf{k}\mathbf{k}'} \left(b^*_{\mathbf{k}} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} + b_{\mathbf{k}'} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} - b^*_{\mathbf{k}} b_{\mathbf{k}'} \right)$$

Then, if we set $V \sum_{\mathbf{k}} b_{\mathbf{k}} \equiv \Delta$, obtain Bogoliubov or Gor'kov Hamiltonian

$$\hat{H} - \mu\hat{N} \simeq \sum_{\mathbf{k}} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \left(\Delta^* c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} + \Delta c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \right) + \frac{|\Delta|^2}{V}$$

Bilinear in fermion operators, $\hat{H} - \mu\hat{N}$ diagonalised by canonical transformation

$$\begin{aligned} c^\dagger_{\mathbf{k}\uparrow} &= u_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}\uparrow} + v_{\mathbf{k}} \gamma_{-\mathbf{k}\downarrow} \\ c_{-\mathbf{k}\downarrow} &= -v_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}\uparrow} + u_{\mathbf{k}} \gamma_{-\mathbf{k}\downarrow} \end{aligned}$$

where anticommutation relation requires $u^2_{\mathbf{k}} + v^2_{\mathbf{k}} = 1$

Taking Δ real (for simplicity), Hamiltonian diagonalised if

$$2\xi_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} + \Delta(v^2_k - u^2_{\mathbf{k}}) = 0$$

With this choice

$$\hat{H} - \mu\hat{N} = \sum_{\mathbf{k}} (\xi_{\mathbf{k}} - \sqrt{\xi^2_{\mathbf{k}} + \Delta^2}) - \frac{\Delta^2}{V} + \sum_{\mathbf{k}\sigma} \sqrt{\xi^2_{\mathbf{k}} + \Delta^2} \gamma^\dagger_{\mathbf{k}\sigma} \gamma_{\mathbf{k}\sigma}$$

Bogoliubov quasi-particle excitations, created by $\gamma^\dagger_{\mathbf{k}\sigma}$, have minimum energy Δ known as the energy gap $\rightsquigarrow$ rigidity of ground state

Ground state wavefunction identified as vacuum state of algebra $\{\gamma_{\mathbf{k}\sigma}, \gamma^\dagger_{\mathbf{k}\sigma}\}$, i.e state which is annihilated by all the quasi-particle operators $\gamma_{\mathbf{k}\sigma}$.

Condition met uniquely by the state

$$|\psi\rangle \equiv \prod_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow}\gamma_{-\mathbf{k}\downarrow}|\Omega\rangle \sim \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}\right)|\Omega\rangle$$

▷ Self-consistency condition

$$\Delta = V\sum_{\mathbf{k}} b_{\mathbf{k}} = V\sum_{\mathbf{k}} \langle\psi|c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}|\psi\rangle = \frac{V\Delta}{2}\sum_{\mathbf{k}} \frac{1-\langle\psi|\sum_{\mathbf{k}\sigma}\gamma^\dagger_\sigma\gamma_{\mathbf{k}\sigma}|\psi\rangle}{\sqrt{\xi_{\mathbf{k}}^2+\Delta^2}}$$

<u>BCS gap equation</u>

At $T=0$, $\langle\psi|\sum_\sigma \gamma^\dagger_{\mathbf{k}\sigma}\gamma_{\mathbf{k}\sigma}|\psi\rangle = 0$ and

$$1 = \frac{V}{2}\sum_{\mathbf{k}} \frac{1}{\sqrt{\xi_{\mathbf{k}}^2+\Delta^2}} = \frac{V\nu(\epsilon_F)}{2}\int_{-\omega_D}^{\omega_D} d\xi \frac{1}{\sqrt{\xi^2+\Delta^2}}$$

if $\omega_D \gg \Delta$, $\Delta \simeq 2\omega_D \exp[-1/\nu(\epsilon_F)V]$

<u>Ground state:</u> In limit $\Delta \to 0$, $v_{\mathbf{k}}^2 \mapsto \theta(\epsilon_F - \epsilon_{\mathbf{k}})$, and the ground state collapses to the filled Fermi sea with chemical potential ϵ_F

As Δ becomes non-zero, states in the vicinity of the Fermi surface rearrange into a condensate of paired states — involves population of single-particle states with energy $\epsilon_{\mathbf{k}} > \epsilon_F$ — but, total energy of ground state is lower than non-interacting Fermi sea however small V

<u>Excitations:</u> It is very important to distinguish quasi-particle states and 'excitations'. Quasi-particle states are the eigenstates of the BCS Hamiltonian with a positive and a negative energy branch. For $\Delta \to 0$, the quasi-particles evolve into ordinary electrons and holes.

By contrast, the energy of excitations (as created by the operators $\gamma^\dagger_{\mathbf{k}\sigma}$) is always positive. An excitation can be either the creation of a quasi-particle at positive energy or the elimination of a quasi-particle (the creation of a quasi-hole) at negative energy. In the ground state, all negative-energy quasi-particle states are filled.

Density of quasi-particle excitations near Fermi surface

$$\rho(\epsilon) = \frac{1}{L^d}\sum_{\mathbf{k}\sigma}\delta(\epsilon - \sqrt{\xi_{\mathbf{k}}+\Delta^2}) = \int d\xi\, \underbrace{\frac{1}{L^d}\sum_{\mathbf{k}\sigma}\delta(\xi-\xi_{\mathbf{k}})}_{\nu(\xi)}\,\delta(\epsilon - \sqrt{\xi+\Delta^2})$$

$$\approx \nu\sum_{s=\pm 1}\int_0^\infty d\xi \frac{\delta\left(\xi - s[\epsilon^2-\Delta^2]^{1/2}\right)}{\left|\frac{\partial[\xi^2+\Delta^2]^{1/2}}{\partial\xi}\right|} = 2\nu\Theta(\epsilon-\Delta)\frac{\epsilon}{(\epsilon^2-\Delta^2)^{1/2}},$$

Spectral weight transferred from Fermi surface to interval $[\Delta,\infty]$

▷ Postpone phenomenology of superconductor until field theory next term